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Estimating the Efficiency of Voting in Big Size Committees

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Abstract In a simple voting committee with a finite number of members, in which each member has a voting weight, the voting rule is defined by the quota (a minimal number of voting weights is required to approve a proposal), and the efficiency of voting in the committee is defined as the ratio of the number of winning coalitions (subsets of the set of members with total voting weights no less than the quota) to the number of all possible coalitions. A straightforward way of calculating the efficiency is based on the full enumeration of all coalitions and testing whether or not they are winning. The enumeration of all coalitions is NP-complete problem (the time required to find the solution grows exponentially with the size of the committee) and is unusable for big size committees. In this paper we are developing three algorithms (two exact and one heuristic) to compute the efficiency for committees with high number of voters within a reasonable timeframe. Algorithms are applied for evaluating the voting efficiency in the Lower House of the Czech Parliament, in the European Parliament and in the Council of Ministers of the EU.

Keywords Efficiency of voting, committee, European Parliament, EU Council of Ministers

JEL classification D71, D72

1. Introduction

While the problem of legitimacy (allocating voting weights to voters) is rather complex and is handled using different concepts of voting power (see e.g. Felsenthal and Machover 1998), the problem of efficiency is relatively simpler: What is the probability that a proposal submitted for voting is approved (i.e., the probability of changing the status quo)? The generally accepted measure of efficiency is the so called Coleman index of the power of a collective to change the status quo (the probability of the appearance of the winning coalition under the assumption of the equal probability of any coalition formation), see Coleman (1971). For empirical studies of different alternative voting rules in the EU Council of Ministers from the standpoint of legitimacy and efficiency see Hosli (2008) and Leech and Aziz (2010).

The main aim of this article is to provide a fast algorithm (running with polynomial time complexity) for the computation of the efficiency of voting systems, as well as some basic analysis of this algorithm. In the context of the European integration process, the minimum number of voters to be investigated is 30. This number of voters

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is hardly reachable by standard algorithms since the total number of coalitions to be investigated is $2^{30} = 1,073,741,824$.

Voting is a way of transforming many individual preferences into one final preference. In this paper, I investigate the efficiency of basic voting systems, also known as a simple weighted majority game, i.e. the probability of a proposal to be approved in a voting process, while the probabilities of rejection are prescribed to each individual voter. The efficiency can be influenced not only by the preferences and probabilities of rejection by each voter, but also by the voting system itself, i.e. by the quota and weights assigned to each individual voter.

In the first part I analyze the efficiency of the simple voting system based on one voting rule while the preference of each voter is given by the rejection of any proposal with 0.5 probability and the acceptance of any proposal with 0.5 probability. This simplification can be used for the selection and tuning of the newly introduced voting systems, where no additional information about the individual preferences are known. It is straightforward that by giving each voter the same weight in the voting, the probability of exactly $k \in \mathbb{N}$ out of $n \in \mathbb{N}$, $k < n$ voters approving the proposal is driven by the binomial probability distribution. The main aim of this paper is to support the creators of voting rules (this can be voting rules in Parliament, general meetings of stockholders as well as in any other organizations, institutions or companies where voting is employed to make one unique decision as a representation of many individual preferences) with some basic knowledge of the efficiency of voting rules. It is important to know which voting system leads to which probability of changing the status quo (approval of the proposal) under the specified preferences of the individuals.

Suppose, we have a set $N = \{1,\dots,n\}$. The cardinality of N is then $|N| = n \in \mathbb{N}$. This set will represent the set of voters (voting bodies), so that each voter is represented by just one index from N. Suppose there is a vector space $\mathbb{V}^n$ above the field of real numbers and the set $\mathbb{S}^n \subset \mathbb{V}^n$ of all vectors $\mathbf{w} = (w_1,\dots,w_n)$, is such that $\sum_{k=1}^{n} w_k = 1$ and for all $k = 1,\dots,n$, $w_k \geq 0$. The set of all real numbers between 0 and 1 (including both) is denoted Λ, i.e. $\Lambda = \{\lambda \in \mathbb{R} : 0 \leq \lambda \leq 1\}$. We call the ordered couples $(\lambda, \mathbf{w}) \in \Lambda \times \mathbb{S}^n$ a committee and the set of all possible committees $\Lambda \times \mathbb{S}^n$ is denoted $\mathbb{M}^n$. The vector $\mathbf{w}$ from $\mathbb{S}^n$ is called a vector of weights and the number λ from Λ is called a quota. A coalition will be called any set of voters, which are represented by a set of indices $Q \subseteq N$, so that $j \in Q \Leftrightarrow$ voter represented by index j accepts the proposal. In other words, a coalition is a set of all the voters who accept the proposal. In the formal definitions it is quite convenient to use the n-dimensional unit cube representation. Suppose there is an n-dimensional unit cube and denote the set of all its vertices $\mathbb{C}^n$, i.e. $\mathbb{C}^n = \{(c_1,\cdots,c_n) : c_i \in \{0,1\}, i \in \{1,...,n\}\}$. The cardinality of $\mathbb{C}^n$ is 2^n. A proposal is approved if and only if the sum of the weights of those voters who accept the proposal is equal to or greater than the quota, i.e.:

$$\sum_{k=1}^{n} w_i c_i \geq \lambda, \tag{1}$$

where $\mathbf{c} = (c_1,\dots,c_n) \in \mathbb{C}^n$ is defined as $c_i = 0$ if the i-th voter rejected the proposal and $c_i = 1$ if he or she approved it.

Hence a coalition can be given by $j \in Q \Leftrightarrow c_j = 1$. Each $\mathbf{c} \in \mathbb{C}^n$ represents just one coalition. The efficiency of a committee $(\lambda, \mathbf{w}) \in \mathbb{M}^n$ is defined as the ratio of all the winning coalitions (coalitions that change the status quo) and all the possible coalitions. This can be formally expressed:

$$\varepsilon(\lambda, \mathbf{w}) := \frac{1}{2^n} \sum_{\mathbf{c} \in \mathbb{C}^n} I\left[\sum_{i=1}^{n} w_i c_i \geq \lambda\right], \tag{2}$$

where the $I[A]$ is the identifier of A, i.e. $I[A] = 1$ if and only if the condition A is true, otherwise it is $I[A] = 0$. The efficiency $\varepsilon(\lambda, \mathbf{w})$ attains only values from the set:

$$\mathscr{Q} := \left\{ \frac{k}{2^n} : k = 1, ..., 2^n \right\}. \tag{3}$$

Since the efficiency function can only attain a finite number of values, it attains its maximum and minimum and it is not continuous. The efficiency is invariant to the order of voters, i.e.: Let Π^n be a set of all permutations of the set $\{1, \ldots, n\}$. Then:

$$(\forall \pi \in \Pi^n) \left(\varepsilon(\lambda, w_1, \ldots, w_n) = \varepsilon\left(\lambda, w_{\pi(1)}, \ldots, w_{\pi(n)}\right)\right). \tag{4}$$

2. Computing efficiency

Standard algorithms that compute the exact efficiency of voting could be used only for committees with a low number of voters, as they have to check all possible coalitions. Computing the efficiency of voting is an NP-complete problem, see Matsui and Matsumi (1999) and Matsui and Matsumi (1998), since adding one more voter approximately doubles the computation time. I have created two algorithms for computing the exact efficiency for general committees. The first one is a simple recursive algorithm (or SRA), the second one is a recursive algorithm with a branch and bound technique employed (or SRB) so as to omit some of the irrelevant committees, once they are known, from further computation. The SRA algorithm seems to be faster for quotas around one half, the SRB algorithm is faster for quotas close to 1 or 0. The reason is clear. The SRB algorithm is faster due to the branch and bound method, which can prune some branches of irrelevant solutions when the quota is close to 1 or 0. On the other hand, the branch and bound technique itself consumes time in verifying the solution relevance in each recursive step. These verifications are very time consuming and can be justified only by saving time via the pruning. However, for quotas around one half, the pruning is not sufficient to prevail over the negative impact of checks on the procedural time.

We can omit computing the exact efficiency and try to estimate it. The question is whether the estimation can be done as precisely as needed in practical applications. In real life we need to be able to estimate the efficiency in percents with reliable certainty. Estimation error can be controlled in probability. I have created a heuristic algorithm (or HRA), which gives not an exact, but a sufficiently precise solution to the efficiency of a simple weighted voting system for committees with a high number of voters.

The HRA algorithm is based on simulating coalitions and checking whether they are winning or not. The number of simulated coalitions will only be a negligible part of the total number of coalitions. Suppose we define a random variable X_i as $X_i = 0$ if and only if the i-th coalition (in the infinite sequence of randomly chosen coalitions from the set of all coalitions) is losing, and $X_i = 1$ if and only if it is winning. The random sequence X_i is i.i.d. as the draws from the set of all coalitions are performed independently[1] and each coalition is generated with the same probability. Then, due to the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = EX_1, \quad [P]-\text{a.s.},$$

because $E|X_1| < \infty$. We know the average converges to a unique number. In addition, it can be shown that this number equals the efficiency. The remaining question is the speed of the convergence, or rather the deviance of the average $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n}$ from the exact efficiency. When $P(X_k \in [a_k, b_k]) = 1$ for $1 \leq k \leq n$, we can use the well-known Hoeffding's inequality:

$$P\left(\left|\sum_{k=1}^{n} X_k - n\varepsilon(\lambda, \mathbf{w})\right| \geq tn\right) \leq 2\exp\left(-\frac{2t^2n^2}{\sum_{k=1}^{n}(b_i - a_i)^2}\right). \tag{5}$$

In our case $a_k = 0$ and $b_k = 1$ for all $1 \leq k \leq n$, and so we get:

$$P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \varepsilon(\lambda, \mathbf{w})\right| \geq t\right) \leq 2\exp\left(-2nt^2\right)$$

and apply it for $t = 0.01$ as a margin of error. We get the probability of an error that is greater than or equal to 0.01 (the efficiency will be different from the exact one by at least 0.01), equal to or lower than $2e^{-0.0002n}$. When using the HRA algorithm we perform 50,000 iterations, and so the probability will be at most 0.0000908.

As previously mentioned, the HRA algorithm is based on independent draws (with repeats) from the set of all possible coalitions and verifying whether they are winning or not. Each draw is done in two steps. In the first step the size of the simulated coalition is randomly chosen from the binomial distribution of the probability, and in the second the voters in this coalition are chosen from the set of all voters performing independent draws from the uniform discrete distribution without repeats (since we do not want to have one voter in any coalition more than once).

In the first step the HRA algorithm has to randomly generate the size of the coalition from the binomial distribution. This procedure has proven to be the most time consuming part of the whole process of efficiency estimation, namely because of the very large numbers of voters, since the binomial distribution requires computing the binomial coefficients. The higher the number of voters, the higher the factorial that

[1] In each step of the simulation some coalitions are generated independently of the previously generated coalitions and hence each coalition can appear more than once in the generated sequence of coalitions.

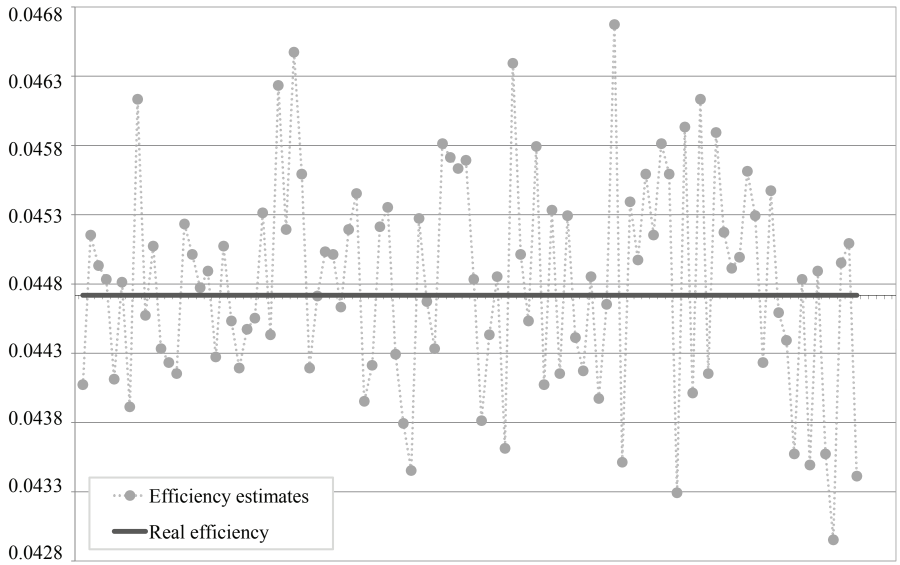

Figure 1. Deviances of simulated efficiencies from their average (100 trials)

needs to be computed. If it were really necessary, we could have not utilized the algorithm for very big committees as it would become time consuming to compute the factorials. Fortunately, computing the factorials is not necessary as we can employ the equality:

$$\sum_{k=0}^{s} \binom{n}{k} p^k (1-p)^{n-k} = F_{2(n-s),2(s+1)} \left[\frac{(s+1)(1-p)}{p(n-s)} \right], \tag{6}$$

where F is the distribution function of the Fisher-Snedecor probability distribution, see Anděl (2004).

The HRA algorithm uses the equality (6) to simulate the size of the coalitions from binomial distribution and thus huge committees can be analyzed.

I have tested the performance of the HRA algorithm and it works well in terms of processing time: It always runs 50,000 iterations, no matter the number of voters, and so it has a constant time complexity. Some simple results showing the variability of the estimations are shown in the following images: In Figure 1, there are 100 estimations of the efficiencies of a simple weighted voting system with 35 voters (with the given distribution of weights and a quota of 0.6; the exact values are not important for now). In this figure, the deviance of each observation from their average is shown.

In Figure 2 the histogram of the estimated efficiencies is shown.

In Table 1, some results of the heuristic algorithm for specified committees are shown.

Table 1. Some estimated efficiencies via the HRA algorithm

Size	Quota and weights	Exact efficiency	Efficiency estimates (5 independent computations)
$n=6$	$\left(0.7; \frac{2}{5}, \frac{1}{6}, \frac{2}{15}, \frac{1}{10}, \frac{1}{20}, \frac{3}{20}\right)$	0.265625	$(0.26895, 0.26395, 0.26418, 0.26622, 0.2654)$
$n=6$	$\left(0.55; \frac{8}{10}, \frac{1}{10}, \frac{1}{10}, 0, 0, 0\right)$	0.5	$(0.49863, 0.50055, 0.50274, 0.49862, 0.50367)$
$n=6$	$\left(0.3; \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6},\right)$	0.890625	$(0.89129, 0.89099, 0.89062, 0.88972, 0.89044)$
$n=9$	$\left(0.9; \frac{22}{50}, \frac{18}{50}, \frac{2}{50}, \frac{3}{50}, \frac{1}{50}, \frac{1}{50}, \frac{1}{50}, \frac{1}{50}, \frac{1}{50}\right)$	0.146484375	$(0, 14378, 0.14113, 0.14161, 0.14357, 0.14357)$
$n=9$	$\left(0.75; \frac{55}{100}, \frac{20}{100}, \frac{8}{100}, \frac{5}{100}, \frac{4}{100}, \frac{3}{100}, \frac{2}{100}, \frac{2}{100}, \frac{1}{100}\right)$	0.279296875	$(0.27947, 0.27958, 0.27860, 0.27865, 0.27983)$
$n=9$	$\left(0.1; \frac{15}{100}, \frac{15}{100}, \frac{15}{100}, \frac{15}{100}, \frac{10}{100}, \frac{10}{100}, \frac{10}{100}, \frac{5}{100}, \frac{5}{100}\right)$	0.994140625	$(0.9939, 0.99457, 0.99414, 0.99433, 0.99421)$
$n=12$	$\left(0.55; \frac{10}{36}, \frac{5}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{2}{36}, \frac{2}{36}, \frac{1}{36}, \frac{1}{36}, \frac{1}{36}, \frac{1}{36}, \frac{1}{36}\right)$	0.423583984	$(0.42403, 0.42444, 0.42324, 0.42445, 0.4247)$
$n=21$	$\left(0.63; \frac{9}{64}, \frac{7}{64}, \frac{6}{64}, \frac{5}{64}, \frac{4}{64}, \frac{4}{64}, \frac{3}{64}, \frac{3}{64}, \frac{3}{64}, \frac{3}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right)$	0.164661884	$(0.16514, 0.16482, 0.1665, 0.16434, 0.16498)$
$n=21$	$\left(0.35; \frac{5}{64}, \frac{5}{64}, \frac{5}{64}, \frac{5}{64}, \frac{5}{64}, \frac{5}{64}, \frac{5}{64}, \frac{4}{64}, \frac{4}{64}, \frac{3}{64}, \frac{3}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{2}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right)$	0.882782936	$(0.88471, 0.88223, 0.88287, 0.88294, 0.88251)$

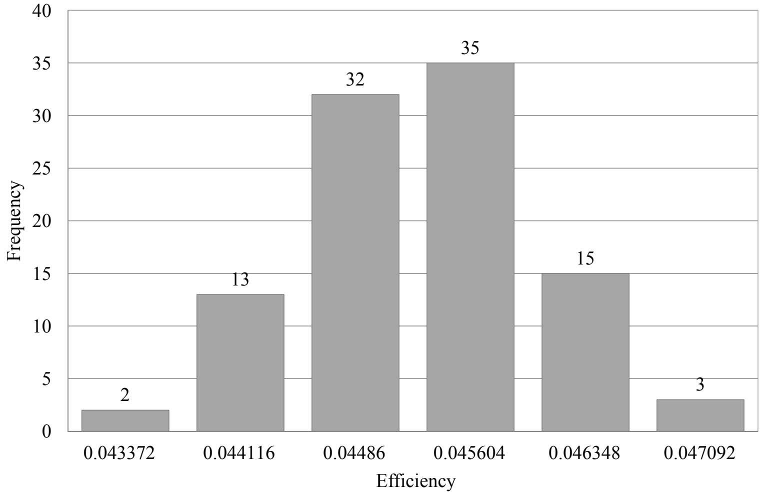

Figure 2. Histogram of simulated efficiencies

3. Efficiency analysis

Lemma 1. *The maximum efficiency for a quota higher than $\frac{1}{2}$ is $\frac{1}{2}$ for any committee size $n \geq 2$.*

Proof. Assume $n \geq 2$. The efficiency of $\frac{1}{2}$ is certainly attained for any quota higher than $\frac{1}{2}$ by assigning $\mathbf{w} = \left(1, \underbrace{0, \ldots, 0}_{(n\text{-}1)\text{-times}}\right)$. When $\sum_{i=1}^{n} w_i c_i > \lambda > \frac{1}{2}$, then $\sum_{i=1}^{n} w_i (1 - c_i) < \lambda$ and $\sum_{i=1}^{n} w_i (1 - c_i)$ is surely not a winning coalition. So for each winning coalition there is at least one losing coalition and so the efficiency can not exceed $\frac{1}{2}$. □

As we show in Lemma 1, no efficiency for a quota above $\frac{1}{2}$ can be above $\frac{1}{2}$, and so we would change any heuristic estimate of the efficiency that is higher than the attainable maximum to $\frac{1}{2}$.

Lemma 2. *Suppose $n \in \mathbb{N}$, $n > 1$ and $\mathbf{w} = \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n\text{-times}}\right)$. If n is odd, then $\varepsilon\left(\frac{1}{2}, \mathbf{w}\right) = \frac{1}{2}$. If n is even, then $\varepsilon\left(\frac{1}{2}, \mathbf{w}\right) = \frac{1}{2}\left(1 + \frac{\binom{n}{\frac{n}{2}}}{2^n}\right)$.*

Proof. The coalitions that are winning are all $\left\lceil \frac{n}{2} \right\rceil$ −element coalitions and all coalitions with more elements.

$$\varepsilon\left(\frac{1}{2},\left(\underbrace{\frac{1}{n},\ldots,\frac{1}{n}}_{n\text{-times}}\right)\right)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{\binom{n}{n-k}}{2^n} \tag{7}$$

For odd n (7) equals $\frac{1}{2}$. For even n (7) equals $\frac{1}{2}\left(1+\frac{\binom{n}{\frac{n}{2}}}{2^n}\right)$. □

Lemma 3. *Suppose $n \in \mathbb{N}$, $n > 1$, $\lambda > \frac{1}{2}$ and $\mathbf{w} = \left(\underbrace{\frac{1}{n},\ldots,\frac{1}{n}}_{n\text{-times}}\right)$. Then:*

$$\varepsilon(\lambda,\mathbf{w})=\sum_{k=\lceil n\lambda\rceil}^{n}\frac{\binom{n}{n-k}}{2^n}. \tag{8}$$

Proof. The coalitions that are winning are all $\lceil n\lambda\rceil$ −element coalitions as well as all coalitions with more elements.

$$\varepsilon\left(\lambda,\left(\underbrace{\frac{1}{n},\ldots,\frac{1}{n}}_{n\text{-times}}\right)\right)=\sum_{k=\lceil n\lambda\rceil}^{n}\frac{\binom{n}{n-k}}{2^n} \tag{9}$$

□

From this simple analysis, we know how to find the weights maximizing the efficiency, when the quota is greater or equal to $\frac{1}{2}$.

The set of rules, which have to be fulfilled in order to approve a proposal, can be larger than just one-element sets. The multi-rule voting system is a system where more than one set of weights is assigned to the voters and more than one quota is employed. The proposal is approved only if accepted by all members of a coalition which is a winning coalition under each of the single rules. These systems are closely studied in Leech et al. (2007). Up to now, we have studied only one-rule systems represented by (1) and here we define multi-rule systems analogously.

A multi-rule weighted voting system is a system in which each proposal is approved if and only if all of the following hold at the same time:

$$\begin{array}{c}\sum_{k=1}^{n} w_{1i}c_i \geq \lambda_1,\\ \vdots\\ \sum_{k=1}^{n} w_{mi}c_i \geq \lambda_m,\end{array} \tag{10}$$

where $\mathbf{c} = (c_1,\ldots,c_n) \in \mathbb{C}^n$ is defined as $c_i = 0$ if the i-th voter rejected the proposal and $c_i = 1$ if he approved it.

Suppose $m \in \mathbb{N}$, $m > 1$. The efficiency of a committee $\left(\lambda_1, \ldots, \lambda_m, \left(\mathbf{w}^1, \ldots, \mathbf{w}^m\right)\right)$, where $\mathbf{w}^i = (w_{i1}, \ldots, w_{in})$, $\sum_{j=1}^n w_{ij} = 1$ for all $i = 1, \ldots, m$ and $w_{ij} \geq 0$ for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$, is defined as the number of all winning coalitions divided by the number of all possible coalitions:

$$\varepsilon\left(\lambda_1, \ldots, \lambda_m, \left(\mathbf{w}^1, \ldots, \mathbf{w}^m\right)\right) = \frac{\sum_{\mathbf{c} \in \mathbb{C}^n} I\left[\sum_{i=1}^n w_{1i} c_i \geq \lambda_1 \wedge \ldots \wedge \sum_{i=1}^n w_{mi} c_i \geq \lambda_m\right]}{2^n}, \tag{11}$$

where the $I[A]$ is the identifier of A, i.e. $I[A] = 1$ if and only if the condition A is true, $I[A] = 0$ otherwise.

While in the case of the one-rule voting system there is only one round of testing for which coalition is winning, in the multi-rule case once we have found the set of coalitions which are winning under the first rule, we continue with this set through to the second rule, third rule and so on. In other words, it is not a problem that would be more complex in terms of time complexity than the one-rule voting system.

The efficiency defined in (11) can only attain values from a finite set. It attains its maximum and minimum and it is not continuous. I have enabled the algorithms SRA, SRB and HRA to treat the multi-rule voting systems simply by making them verify more than just one rule when verifying whether a given coalition is winning. I applied the HRA algorithm to compute the efficiencies of the qualified majority voting process of the Council of Ministers of the EU (a good example of the multi-rule voting procedure) under the Treaty of Nice and under the Lisbon Treaty. I also briefly analyze some of the rules separately and their contribution to the total efficiency. This approach shows the level of redundancy of the particular rules.

4. Applications

4.1 Czech Lower House of the Parliament in 2006

The Czech Parliamentary voting system used for the Lower House of the Parliament is based on the law *Zákon 247/1995 Sb., O volbách do Parlamentu České republiky a o změně a doplnění některých dalších zákonů* (1995). It is a two-stage voting system. In the first stage, mandates are allocated to regional districts defined by the law using a combination of Hagenbach-Bischoff's quota and the method of maximal reminder. Then the limit of 5% is applied to reject all the parties which do not have over 5% of all the valid votes at the national level from further allocation. In the second stage the d'Hondt method is used to assign mandates to the parties which have not been rejected. The second stage is performed in each district separately. There are 200 mandates in the Lower House of the Parliament.

The regular approval process is based on the majority rule.[2] This corresponds to a committee with a quota equal to $q = 101/200$ and weights equal to the shares of mandates assigned to each party in the elections.[3] For the constitutional proposals

[2] We only analyze the absolute majority rule since the simple majority can be represented by many committees with different weights according to how many members of each party are momentarily absent.

[3] This committee is based on the assumption that each member voting is always the same as all the other members of the same party.

approval process the quota is different $q = 3/5$.

First we compute the efficiency of the absolute majority committee resulting in the 2006 Parliamentary elections and we look at how different the political division of the Lower House of the Parliament would have to be in order to increase the efficiency to its maximum. Then we compute the efficiency for the constitutional approval process.

Table 2. The results of the 2006 Parliamentary elections in the Czech Republic

Party (name in 2006)	Votes (in %)	Deputies
ODS	35.38%	81
ČSSD	32.32%	74
KSČM	12.81%	26
KDU-ČSL	7.22%	13
Strana zelených	6.29%	6

The results of the 2006 Czech Parliamentary elections are shown in the Table 2. The efficiency of the absolute majority committee given by the results of 2006 elections equals 0.46875. This result can be obtained via the SRA or SRB algorithms since the number of parties is low. We know from the Lemma 1 that the maximum possible efficiency for quotas above $1/2$ equals $1/2$. Hence we search for the closest vector of weights with the efficiency $1/2$. The resulting set of weights for the quota $101/200$ (corresponding to the set of parties (ODS, ČSSD, KSČM, KDU-ČSL, SZ) is $(0.40285, 0.3689, 0.1361, 0.06305, 0.0291)$ compared to the observed vector of weights given to the parties in the 2006 elections $(0.405, 0.37, 0.13, 0.065, 0.03)$. The result is somewhat surprising because it is clearly not that far from the actual set of weights (the distance in L^2-norm equals 0.00690326, which is about 0.8% of the maximal distance in L^2-norm for 5-member voting systems). The numbers of deputies assigned to each party would have to be modified very little in order to achieve the maximal efficiency from $(81, 74, 26, 13, 6)$ to $(80, 74, 27, 13, 6)$. Regardless, we have to keep in mind that this analysis is purely theoretical and neglects the political positions of each of the parties.

In the case of a constitutional majority with a quota equal to $3/5$ the efficiency is 0.25.

4.2 The European Parliament in 2009

At the end of 2009 there were 736 members representing 27 EU countries in the EU Parliament, which was governed by the Maastricht Treaty. The numbers of mandates assigned to each country are shown in the Table 3 and follows the degressive proportionality principle. Assume we want to revise the weights assigned to each particular country in order to increase the efficiency. To do this we have to take into account the proposals for which all the countries are voting homogeneously.[4]

We assume they follow the procedure of absolute majority voting. The efficiency of voting in the European Parliament for an absolute majority voting procedure under the

[4] For most of the real proposals in the EU Parliament voting about proposals is based on political viewpoints rather than nationality.

Table 3. Portions and numbers of mandates assigned to each country in the European Parliament via the Maastricht Treaty

Country	Weight (in %)	Members	Country	Weight (in %)	Members
Germany	13.5%	99	France	9.8%	72
Italy	9.8%	72	Belgium	3.0%	22
Netherlands	3.4%	25	Luxembourg	0.8%	6
Great Britain	9.8%	72	Denmark	1.8%	13
Ireland	1.6%	12	Greece	3.0%	22
Spain	6.8%	50	Portugal	3.0%	22
Sweden	2.5%	18	Austria	2.3%	17
Finland	1.8%	13	Poland	6.8%	50
Czech Republic	2.7%	22	Hungary	2.7%	22
Slovakia	1.8%	13	Slovenia	1.0%	7
Latvia	1.1%	8	Lithuania	1.6%	12
Cyprus	0.8%	6	Estonia	0.8%	6
Malta	0.7%	5	Romania	4.5%	33
Bulgaria	2.3%	17			

assumption of the probability of acceptance equal to 0.5 is 0.49798815. This is very close to the maximal possible efficiency, which is 0.5. Therefore, there is no practical need to change the weights to achieve a higher a priori efficiency of the committee under the assumption of a country-homogeneous proposal. The efficiencies computed for the European Parliament absolute majority voting system work under the assumption of country-homogeneous voting as shown in Table 4.

Table 4. The efficiency of the absolute majority of the 2009 European Parliament given the probabilities of acceptance and country-homogeneous voting

p	0	0.05	0.1	0.15	0.2
ε	0%	0%	0.004%	0.104%	0.532%
p	0.25	0.3	0.35	0.4	0.45
ε	1.974%	5.422%	12.130%	21.822%	35.188%
p	0.5	0.55	0.6	0.65	0.7
ε	49.8500%	64.564%	77.534%	88.162%	94.348%
p	0.75	0.8	0.85	0.9	0.95
ε	97.888%	99.368%	99.872%	99.998%	100.000%

At the end of 2009 in the European Parliament there were 6 political groups, in which 709 members were organized, and 27 independent members not belonging to any of these 6 groups. The groups are listed in Table 5, where the independent members

Table 5. The political structure of the European Parliament as of the end of 2009

EPPD (Conservative/Christian Democrat)	265
ECR (Conservatives only)	54
S&D (Social Democrats)	184
EUL/NGL (Communists/Far-left)	35
ALDE (Liberal/Centrist)	85
G/EFA (Greens/Regionalists)	55
NI (Independents)	27
EFD (Eurosceptics)	31

Table 6. The efficiency of the absolute majority of the 2009 European Parliament given the probabilities of acceptance and party-homogeneous voting

p	0	0.05	0.1	0.15	0.2
ε	0%	0.308%	1.524%	3.532%	6.762%
p	0.25	0.3	0.35	0.4	0.45
ε	10.936%	16.824%	23.282%	31.548%	39.708%
p	0.5	0.55	0.6	0.65	0.7
ε	49.800%	59.392%	68.220%	76.342%	82.952%
p	0.75	0.8	0.85	0.9	0.95
ε	89.074%	93.332%	96.400%	98.536%	99.676%

are assigned to a group of Independents. However, each member of the Independents is treated as one single political party for the efficiency computation.

When we assume the absolute majority procedure and only consider the proposals for which the voting is party-homogeneous (all members within one political group vote the same way), we end up with a different committee. I have analyzed the efficiencies of this voting system for different probabilities of acceptance. The results are shown in Table 6.

When we look at the graph in Figure 3, we can conclude:

(i) When the proposal is in the category of country-homogeneous proposals (each country votes as one individual), the probability it will be approved is higher than that of a proposal from the category of party-homogeneous proposals, but only for probabilities of acceptance higher than approximately one half.

(ii) When the proposal is in the category of country-homogeneous proposals (each country votes as one individual), the probability it will be approved is lower than that of a proposal from the category of party-homogeneous proposals, but only for probabilities of acceptance lower than approximately one half.

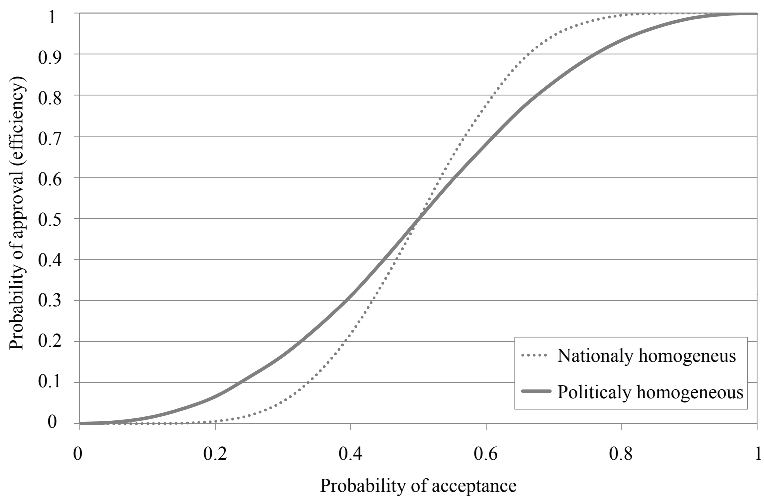

Figure 3. The efficiency with respect to the probability of acceptance within the 2009 European Parliament absolute majority procedure

4.3 The Council of Ministers of the EU in 2009

In this part, I study the efficiency of qualified majority voting in the Council of Ministers of the EU under the Treaty of Nice and the Treaty of Lisbon. The Council of Ministers of the EU has 27 Members States. Their weights in the qualified majority voting rule and the most recent available population estimation of each of the Member States are shown in Table 7.

The multi-rule voting system applied to most of the proposals that are subject to voting in The Council of Ministers of the EU under the Treaty of Nice is the qualified majority voting, which is given by the following three rules:

(i) The sum of the weights of the approving states has to be at least 255 out of 345 to accept a proposal.

(ii) The number of the approving states has to be at least 14 out of 27 to accept a proposal.

(iii) The population of the approving states has to make up at least 62% of the total population of the EU to accept a proposal.

The efficiency estimation (given the probability of acceptance is 0.5 for all members) of the qualified majority multi-rule voting system in The Council of Ministers of the EU under the Treaty of Nice equals approximately 1.9%. The estimated efficiencies (probabilities of approving a proposal) for the different probabilities of acceptance of a single voter are shown in the Table 8 and Figure 5.

Table 7. The population and weights assigned to states in qualified majority voting in the Council of Ministers of the EU via the Treaty of Nice

State	Weight	Population (mil.)	State	Weight	Population (mil.)
Germany	29	82.3	Italy	29	59.7
France	29	64.5	United Kingdom	29	61.0
Spain	27	45.2	Poland	27	38.1
Romania	14	22.3	The Netherlands	13	16.4
Belgium	12	10.6	Czech Republic	12	10.4
Greece	12	11.2	Hungary	12	10.0
Portugal	12	10.6	Austria	10	8.3
Sweden	10	9.2	Bulgaria	10	7.7
Denmark	7	5.5	Denmark	7	5.5
Ireland	7	4.4	Lithuania	7	3.4
Slovakia	7	5.4	Finland	7	5.3
Cyprus	4	0.8	Estonia	4	1.4
Latvia	4	2.3	Luxembourg	4	0.5
Slovenia	4	2.0	Malta	3	0.4

Table 8. The efficiency of the qualified majority via the Treaty of Nice for the given probabilities of acceptance

p	0	0.05	0.1	0.15	0.2
ε	0%	0%	0%	0%	0%
p	0.25	0.3	0.35	0.4	0.45
ε	0.0005%	0.0075%	0.034%	0.173%	0.600%
p	0.5	0.55	0.6	0.65	0.7
ε	1.984%	5.288%	11.587%	22.520%	38.089%
p	0.75	0.8	0.85	0.9	0.95
ε	56.9167%	75.4645%	89.856%	97.452%	99.828%

We know, the total number of coalitions equals $2^{27} = 134,217,728$. In Table 9 and in Figure 4, the efficiencies for the modified qualified majority of The Council of Ministers of the EU under the Treaty of Nice are shown. There are two modifications: The first one leaves out the rule of population (denoted ε_1) and the second leaves out the rule of artificial weights (denoted ε_2).

As can be seen in the graph, the impact of the rule of population in the qualified majority voting under the Treaty of Nice is almost redundant. The only reason for it to be applied is for protection against a radical change in the population distribution among member states. Interestingly, leaving out the rule of artificial weights would significantly increase the probability of acceptance. On the other hand, this would mean overwhelming the small states. This result seems to be quite intuitive since the concept of fairness and the concept of efficiency are often in contradiction.

Table 9. The impact of distinct rules on the efficiency of the qualified majority via the Treaty of Nice for the given probabilities of acceptance

p	0	0.05	0.1	0.15	0.2
ε_1	0%	0%	0%	0%	0%
ε_2	0%	0%	0%	0%	0.006%
p	0.25	0.3	0.35	0.4	0.45
ε_1	0%	0.008%	0.038%	0.170%	0.696%
ε_2	0.038%	0.358%	1.460%	4.386%	10.272%
p	0.5	0.55	0.6	0.65	0.7
ε_1	1.970%	5.162%	11.700%	22.324%	37.918%
ε_2	18.804%	31.156%	44.764%	59.068%	71.582%
p	0.75	0.8	0.85	0.9	0.95
ε_1	56.736%	75.772%	89.9313%	97.456%	99.856%
ε_2	82.852%	91.408%	96.330%	99.120%	99.890%

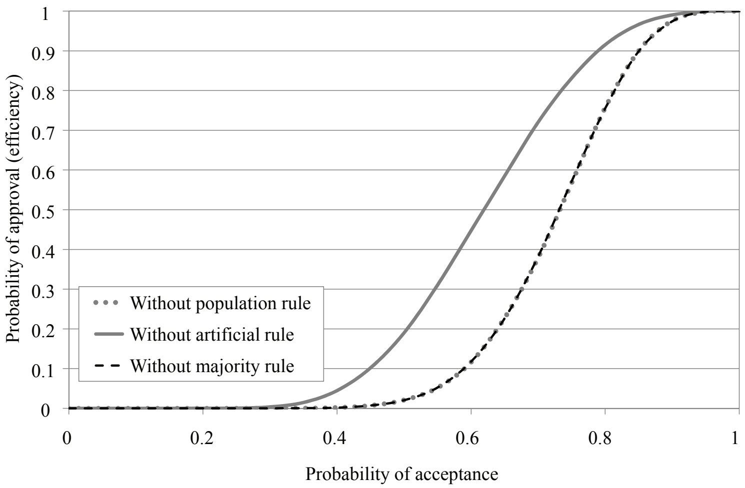

Figure 4. The impact of the distinct rules on efficiency within the procedure of the qualified majority via the Treaty of Nice

Under the Lisbon Treaty the procedure of the qualified majority can be simplified to just two rules:

(i) At least 55% of all the states have to accept the proposal.

(ii) The population of the approving states has to make up at least 65% of the total population of the EU to accept a proposal.

The efficiency estimation (given that the probability of acceptance is 0.5 for all members) of the qualified majority multi-rule voting system in The Council of Ministers of the EU under the Lisbon Treaty equals approximately 12.7%. The estimated efficiencies (probabilities of approving a proposal) for the different probabilities of the acceptance of a single voter are shown in the Table 10 and Figure 5.

Table 10. The efficiency of the qualified majority via the Treaty of Lisbon for the given probabilities of acceptance

p	0	0.05	0.1	0.15	0.2
ε	0%	0%	0%	0%	0.002%
p	0.25	0.3	0.35	0.4	0.45
ε	0.030%	0.128%	0.676%	2.296%	6.090%
p	0.5	0.55	0.6	0.65	0.7
ε	12.664%	23.326%	36.248%	50.034%	64.262%
p	0.75	0.8	0.85	0.9	0.95
ε	78.028%	87.642%	94.768%	98.534%	99.852%

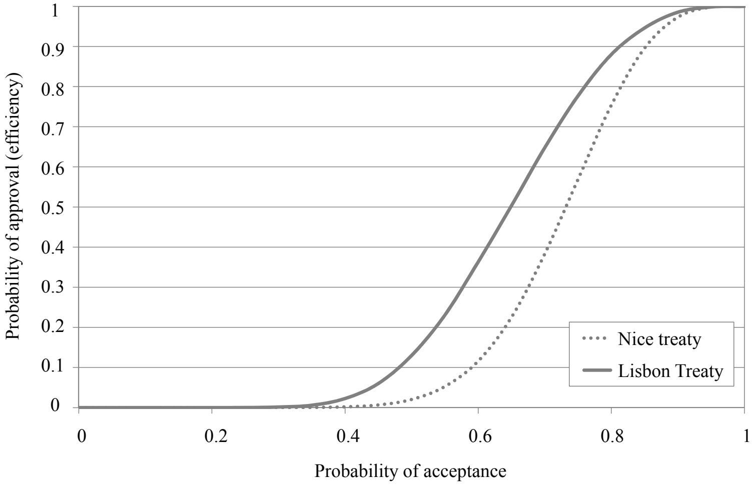

Figure 5. The efficiency with respect to probability of acceptance within the procedure of qualified majority via the Treaty of Nice and the Lisbon Treaty

In the Table 11 and Figure 6, the efficiencies for the modified qualified majority of The Council of Ministers of the EU under the Lisbon Treaty are shown. There are two modifications: The first one leaves out the rule of population (denoted ε_1) and the second one leaves out the rule of the number of states (denoted ε_2).

Table 11. The impact of distinct rules on the efficiency of the qualified majority via the Lisbon Treaty for the given probabilities of acceptance

p	0	0.05	0.1	0.15	0.2
ε_1	0%	0%	0%	0%	0.004%
ε_2	0%	0%	0%	0.028%	0.096%
p	0.25	0.3	0.35	0.4	0.45
ε_1	0.074%	0.422%	2.324%	7.458%	18.138%
ε_2	0.336%	1.060%	2.514%	5.250%	10.198%
p	0.5	0.55	0.6	0.65	0.7
ε_1	34.982%	55.706%	75.146%	88.650%	96.442%
ε_2	16.876%	26.764%	38.508%	51.716%	65.286%
p	0.75	0.8	0.85	0.9	0.95
ε_1	99.274%	99.908%	99.998%	100.000%	100.000%
ε_2	78.056%	87.620%	94.786%	98.578%	99.842%

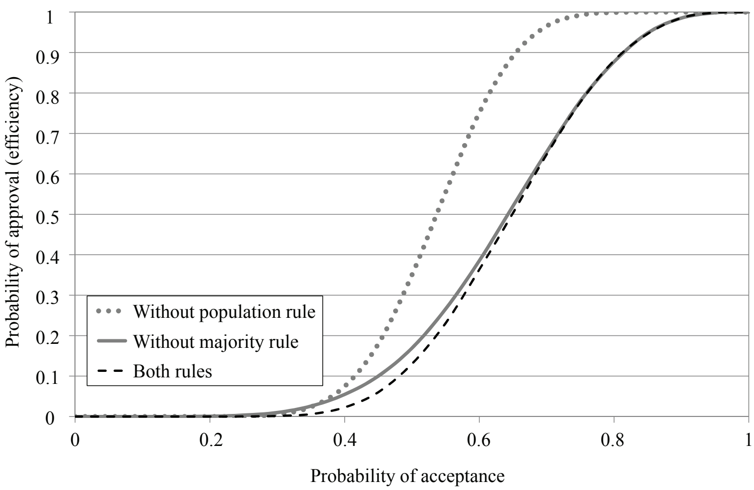

Figure 6. The impact of the distinct rules on efficiency within the procedure of a qualified majority via the Lisbon Treaty

As can be seen in the graph, the impact of the rule of the number of states in the qualified majority voting under the Lisbon Treaty is almost redundant. On the other hand leaving out the population rule would significantly increase the probability of acceptance.

If we compare the qualified majority under the Treaty of Nice and under the Lisbon Treaty, we end up with an expected result of a significantly higher efficiency under the Lisbon Treaty since the smaller states lose part of their ability to block proposals.

5. Conclusion

I have provided two exact and one heuristic algorithm for efficiency computation and some basic theoretical analysis of the efficiency function as a function of quota and weights, focusing mainly on studying the quota and deriving some characteristics of this function with respect to the quota. Then I applied this knowledge to some practical problems. At first finding how far is the outcome of the 2006 Parliamentary elections for the Czech Lower House of the Parliament from an outcome that would represent a vector of weights for which the maximal efficiency would be attained. I have found out that the situation in 2009 in the Czech Lower House of the Parliament was very close to what would represent a maximal a priori efficiency (as defined in the article) for the approval of ordinary proposals and the outcome is relatively far from the maximum a priori efficiency for the approval of constitutional proposals. The quota for constitutional proposal approval seems to be sufficient as a constitution safety guarantee because the closest vector of weights that represents a maximal a priori efficiency is only for two parties being elected to the Lower House of the Parliament, each with the weight 0.5.

Then I computed the efficiency of the voting system of the former European Parliament, assuming the simple majority quota for both country and political dimensions. Finally the heuristic algorithm was used for comparing the efficiency of voting under the qualified majority rule in the Council of Ministers of the EU under the Treaty of Nice and the Lisbon Treaty. Then each of these two procedures were analyzed separately to verify the distinct rules and their impact on the final efficiency. We could see the population rule in the qualified majority under the Treaty of Nice is redundant. However the most influential rule of artificially assigned weights to each state was abandoned in the procedure under the Lisbon Treaty and hence the efficiency of the voting increased significantly.

The developed heuristic algorithm generally runs at a constant time no matter the number of voters, but with a limited preciseness. It can be used, with slight changes, for estimations of most of the power indices. The generalized power indices (see Aleskerov 2006) would require deeper changes.

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References

Aleskerov, F. (2006). Power Indices Taking into Account Agents Preferences. Mathematics and Democracy, Studies in Choice and Welfare, 1-18.

Anděl, J. (2004). *Statistické metody*. Prague, Matfyzpress.

Coleman, J. S. (1971). Control of Collectivities and the Power of a Collectivity to Act. In Lieberman, B. (ed.), Social Choice. New York, Gordon and Breach, 269-300.

Felsenthal, D. S. and Machover, M. (1998). *The Measurement of Voting Power: Theory and Practice*. Cheltenham, Edward Elgar.

Hosli, M. 2008. Council Decision Rules and European Constitutional Design. *AUCO Czech Economic Review*, 2(1), 76–96.

Leech, D. and Aziz, H. (2010). The Double Majority Voting Rule of the EU Reform Treaty as a Democratic Ideal for an Enlarging Union: an Appraisal Using Voting Power Analysis. In Cichocki, M. and Zyczkowski, K. (eds.), *Institutional Design and Voting Power in the European Union*. Farnham, Ashgate, 59–74.

Leech, D., Aziz, H. and Paterson, M. (2007). Combinatorial and Computational Aspects of Multiple Weighted Voting games. Warwick, University of Warwick, Department of Economics, Working Paper No. 823.

Matsui, T. and Matsumi, Y. (1998). NP-completeness for Calculating Power Indices of Weighted Majority Games. *Theoretical Computer Science*, 263, 98–101.

Matsui, T. and Matsumi, Y. (1999). A Survey of Algorithms for Calculating Power Indices of Weighted Majority Games. Journal of the Operations Research Society of Japan, 43, 71–86.

Zákon 247/1995 Sb., O volbách do Parlamentu České republiky a o změně a doplnění některých dalších zákonů (1995).